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# ON THE VACUUM STRUCTURE IN THE COULOMB AND LANDAU GAUGES

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Vacuum structure in the SU($N$) Coulomb and Landau gauges is studied by using the methods of harmonic maps. A systematic way for solving the Gribov vacuum copy equation is presented and many examples are discussed in both the Coulomb and Landau gauges as applications of the method. Finally, the physical interpretation of Gribov ambiguities is shortly reviewed from a topological point of view.

## 1. Introduction

In this article I present a systematic method of solving the Gribov vacuum copy equation in the SU($N$) Coulomb and Landau gauges and discuss some properties of the allowed field configurations.

The problem of Gribov ambiguities goes back to 1977 when Gribov [1, 2] observed that the Coulomb gauge-fixing condition does not uniquely fix the gauge but allows some remaining gauge freedom. After Gribov's observation an extensive literature has grown on the subject but so far an acceptable physical interpretation has not been given. Many configurations attainable in the vacuum sector are known [1–11]; the approach has been in terms of some more or less successful ansätze and no profound relationships between different ansätze have been found.

Mathematically the vacuum sector solutions of the gauge copy equation are harmonic maps between the manifolds $\mathrm{R}^k$ and SU($N$) [9], and in the present article I will use this observation to develop a systematic method for solving the guage copy equation in the vacuum sector. This method is based on finding certain harmonic trace maps between some submanifolds of $\mathrm{R}^k$ and SU($N$), and solutions to the original equation are found by immersing these trace maps. The use of trace maps is motivated by the present state of the theory of harmonic maps; there are practically no results to be applied in the problem of harmonic maps from a non-compact manifold to a compact one. But by using harmonic trace maps defined on compact manifolds one can apply the results in the theory of harmonic maps to the present

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problem. However, the lack of any reasonable classification and the divergence of the energy functional makes it very difficult to find global results. Thus, uniqueness and existence results concerning physically reasonable configurations have not yet been found.

The contents of this article is the following. In sect. 2 the notation is presented and some general results and properties of the gauge fixing are discussed. In sect. 3 the method of immersion is presented and the rest of this article describes its use: in sect. 4. SU(2) vacuum structure is discussed and in sect. 5. I present some properties of the general SU($N$) gauge group and as applications work out some cases in the SU(3) gauge group. In sect. 6. I review some physical interpretations.

## 2. The Gribov ambiguity

In order to set the stage and to fix the conventions recall the euclidean pure gauge field theory defined by the lagrangian

$$\mathcal{L} = \tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \,,$$

where $F^a_{\mu\nu}$, the field strength tensor, is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \,.$$

The $f^{abc}$'s are the structure constants of the invariant vector fields on the group manifold SU($N$). At the identity $e \in \mathrm{SU}(N)$ I choose as basis for the $\underline{\mathrm{su}}(N)$ Lie algebra a set of $N^2-1$ traceless hermitian matrices $T^a$, $a = 1, \ldots, N^2-1$ satisfying the Cartan inner product relation

$$\mathrm{Tr}\,\{T^a T^b\} = 2\delta^{ab} \,.$$

The normalization chosen means that for SU(2) the $T^a$'s are the Pauli matrices and for SU(3) they are the Gell-Mann matrices. The vacuum sector of the theory is defined by the identical vanishing of the field strength tensor $F^a_{\mu\nu}$, which is equivalent to saying that the Lie algebra valued gauge field $A_\mu$ has the form

$$A_\mu = A^a_\mu \frac{T^a}{2} = -\frac{i}{g} (\partial_\mu \omega) \omega^{-1} \,. \tag{1}$$

Here $\omega$, the gauge transform matrix, is a map from the euclidean space $\mathrm{R}^k$ to the gauge group SU($N$) represented by a unitary matrix.

Gribov discovered the following ambiguity in the Coulomb gauge formulation of the SU($N$) gauge theory: the Coulomb gauge-fixing condition

$$\partial_i A_i = 0 \tag{2}$$

does not uniquely fix the gauge but in a chosen gauge-field orbit there are several gauge-field configurations satisfying the condition (2). This lack of uniqueness gives the familiar technical problems in the quantization of the theory, and one might think to circumvent these difficulties by using a gauge-fixing condition that does not allow ambiguities. In fact, there are at least two gauge-fixing conditions that avoid the Gribov ambiguities [12, 13]. These gauge-fixing conditions, however, have their own difficulties like selecting out preferred directions and breaking down translational invariance. Moreover, they are cumbersome to use in practical calculations and the compactification of the spatial part of four-space is not allowed. Indeed, the Gribov ambiguity seems to be more than just a technical problem: it is always present in a compactified space [14, 15] implying that the quantized free gauge theory cannot be formulated in terms of $N^2-1$ interacting gluons only. This means that in a compactified space the spectrum of states does not contain free gluon states, which can be interpreted as color confinement. This approach has so far not been put in a satisfactory form and the possible confining effects of Gribov ambiguities have not been declared [16]. However, intensive work has been pursued in formulating in all continuous gauges, using the degeneracy, a similar vacuum structure which we have in the temporal and axial gauges. This idea was introduced by Wadia and Yoneya [17] and the approach seems to be very promising, at least in the Coulomb gauge: the BPST instanton, truly a gauge-invariant object, tunnels between the two Gribov vacua in the Coulomb gauge [18], and, allowing discontinuities in the time evolution of the gauge fields, there seems to be a rich tunneling picture [19–21]. Introduction of meron like configurations seems to restore the symmetry of the vacuum and brings the theory to the confining phase, along the ideas of the confinement mechanism by Callan, Dashen and Gross [22, 23]. So far this program has not been completed. This is partly due to the inadequate knowledge of the possible vacuum configurations attainable in the Coulomb gauge, and the purpose of this article is to develop a systematic method for searching for the relevant configurations in the Coulomb and Landau gauges.

Let us now investigate the Coulomb and Landau gauge-field configurations in the vacuum sector. The mathematical properties of these gauges will turn out to be almost identical and so I will adopt the following convention: in the Coulomb gauge the greek summation index will run over the values 1, 2, 3 and in the Landau gauge the allowed values will be 1, 2, 3, 4. Most of the analysis can also be applied to the classical theory of the principal chiral model [24] in the euclidean two-space $\mathrm{R}^2$, and to include this theory into the formalism I will assume that the greek summation index can also be attached to the values 1, 2 only. In what follows it should always be clear which of the dimensionalities are allowed.

In the vacuum sector the divergence condition $\partial_\mu A_\mu = 0$ can be put into the form

$$\partial^2\omega + (\partial_\mu\omega \cdot \partial_\mu\omega^{-1})\omega = 0\,, \tag{3}$$

which can be shown to be the Euler-Lagrange equation for the energy functional [1]

$$E(\omega) = \frac{1}{2}\int_{\mathrm{R}^k} \mathrm{Tr}\,\{\partial_\mu\omega \cdot \partial_\mu\omega^{-1}\}\,\mathrm{d}^k x\,. \tag{4}$$

The extremals of the energy functional are customarily called harmonic maps, and for further analysis of the energy functional see [9] and the references therein.

The energy functional (4) can be interpreted as a definition of a chiral field theory of interacting pion fields, where the fields are the matrix elements of the gauge transform matrix $\omega$. The energy-momentum stress tensor $T_{\mu\nu}$ of the chiral theory is

$$T_{\mu\nu} = \mathrm{Tr}\,\{\partial_\mu\omega\,\partial_\nu\omega^{-1}\} + \mathrm{Tr}\,\{\partial_\nu\omega\,\partial_\mu\omega^{-1}\} - \delta_{\mu\nu}\,\mathrm{Tr}\,\{\partial_\alpha\omega\,\partial_\alpha\omega^{-1}\}\,,$$

and it is conserved, $\partial_\mu T_{\mu\nu} = 0$, as a consequence of the equations of motion. By introducing local coordinates $\{\omega^i\}$ on the group manifold SU($N$) and defining $g_{ik}$ to be the metric tensor on SU($N$) the energy-momentum stress tensor $T_{\mu\nu}$ can be given the local form

$$T_{\mu\nu} = 2g_{ik}\frac{\partial\omega^i}{\partial x_\mu}\frac{\partial\omega^k}{\partial x_\nu} - \delta_{\mu\nu}g_{ik}\frac{\partial\omega^i}{\partial x_\alpha}\frac{\partial\omega^k}{\partial x_\alpha}\,.$$

In the principal chiral model the properties of the energy-momentum stress tensor can be used to find new information of the classical solutions: by requiring finiteness of the energy functional (4) one gets the equations [25]

$$g_{ik}\frac{\partial\omega^i}{\partial x}\frac{\partial\omega^k}{\partial x} = g_{ik}\frac{\partial\omega^i}{\partial y}\frac{\partial\omega^k}{\partial y}\,, \qquad g_{ik}\frac{\partial\omega^i}{\partial x}\frac{\partial\omega^k}{\partial y} = 0\,.$$

Mathematically these equations say that the classical solutions are conformal maps, so they will preserve the angles.

In two dimensions the energy functional (4) is invariant under conformal transforms of the domain [26]. The conformal equivalence $\mathrm{R}^2 \cup \{\infty\} \simeq \mathrm{S}^2$ can be used for finite-energy solutions to formulate the theory on the sphere $\mathrm{S}^2$. This means that these classical solutions are harmonic maps from the sphere $\mathrm{S}^2$ to the gauge group SU($N$). Now, supposing that $\omega_1$ and $\omega_2$ both solve the Euler–Lagrange equation (3), the condition for the product $\omega_1\omega_2$ to be a solution can be expressed as a commutator relation:

$$[\omega_1^{-1}\cdot\partial_\mu\omega_1, \partial_\mu\omega_2\cdot\omega_2^{-1}]\,. \tag{5}$$

If one splits $\mathrm{R}^4 = \mathrm{R}^2 \times \mathrm{R}^2$, one notices that the solutions of the principal chiral model give solutions to this commutator relation provided that the two solutions $\omega_1$ and $\omega_2$ live in different $\mathrm{R}^2$ subspaces. Hence the solutions of the principal chiral model give vacuum copies in the Coulomb and Landau gauges.

Let us now look at the more general boundary conditions at infinity. In the WBC [27] case,

$$\lim_{r\to\infty}\omega(r,\Omega) = \omega_0(\Omega)\,. \tag{6}$$

$\omega_0$ is a well-defined map from the euclidean sphere $\mathrm{S}_\infty^{k-1}$ at infinity to the gauge group

SU($N$). By taking the limit $r\to\infty$ in the Euler–Lagrange equation (3) one gets

$$L^2\omega_0=(\partial_T\omega_0\partial_T\omega_0^{-1})\omega_0\,. \tag{7}$$

Here $L^2$ is the Laplace–Beltrami operator on $S_\infty^{k-1}$ and $\partial_T$ is the transverse part of the gradient. Eq. (7) says that $\omega_0$ is a harmonic map from the sphere $S_\infty^{k-1}$ to the gauge group SU($N$). In the case of $S_\infty^1$ these harmonic maps are the closed geodesics on the manifold SU($N$). In the $S_\infty^3$ case we have the non-trivial homotopy structure

$$\Pi_3[\mathrm{SU}(N)]\simeq Z\,, \tag{8}$$

and thus it might be interesting to know the homotopy classes of the allowed maps $\omega_0$. The result is that these $\omega_0$'s all belong to the trivial homotopy class in the classification (8). This can be seen as follows: the vacuum sector is defined by the identical vanishing of the field strength tensor $F_{\mu\nu}$ which implies that the topological charge, defined by

$$\nu=\frac{g^2}{64\pi^2}\varepsilon_{\alpha\beta\gamma\delta}\int_{\mathrm{R}^4}\mathrm{d}^4xF^a_{\alpha\beta}F^a_{\gamma\delta}$$

must vanish.

By changing this integral to a surface integral, one gets

$$\nu=\frac{1}{24\pi^2}\oint_{\mathrm{R}\to\infty}\mathrm{d}\sigma_\mu\varepsilon_{\mu\alpha\beta\gamma}\,\mathrm{Tr}\,\{\omega\partial_\alpha\omega^{-1}\cdot\omega\partial_\beta\omega^{-1}\cdot\omega\partial_\gamma\omega^{-1}\}\,. \tag{9}$$

By substituting the asymptotic map $\omega_0$ in this integral and noticing that expression (9) gives the Brouwer degree of the map $\omega_0$, $S_\infty^3\to\mathrm{SU}(N)$, one finds that $\nu$ vanishes and therefore $\omega_0$ belongs to the trivial homotopy class.

In the case of the SBC boundary condition [27] corresponding to the one-point compactification of the euclidean $k$-space $\mathrm{R}^k\cup\{\infty\}\simeq \mathrm{S}^k$, one can use the real analyticity of the solutions which allows one to perform a Laurent expansion in powers of $1/r$ at large distances:

$$\omega(r,\Omega)=\sum_{n=0}^{\infty}\omega_n(\Omega)\left(\frac{1}{r}\right)^n\,. \tag{10}$$

In this expansion $\omega_0$ is always non-vanishing. Supposing that the first non-trivial term in the powers of $1/r$ is $\omega_n(\Omega)\cdot(1/r)^n$ and by substituting the Laurent expansion (10) into the Euler-Lagrange equation (3) and collecting terms of equal powers in $1/r$ one gets the equation

$$L^2\omega_n=n(n-k+2)\omega_n\,,$$

which means that the elements of the matrix $\omega_n$ are harmonic polynomials of a common degree. But the eigenvalues of the Laplace-Beltrami operator $L^2$ on the sphere $S^{k-1}$ are of the form $j(j+k-2)$ and by substituting $n=j+k-2$ one concludes that in the principal chiral model, corresponding to $k=2$, and in the

Coulomb gauge, $k=3$, all integer values $n \geq 1$ are allowed. But in the Landau gauge, $k=4$, $n$ must be larger than 2. Recalling that the expansion (10) is a stationary point of the energy functional (4) one observes, after power counting, that the energy functional must be finite. If one scales $x_\mu \rightarrow \lambda x_\mu$ in the energy functional one gets

$$E_\lambda(\omega)=\lambda^{2-k}E(\omega)\,,$$

and stationarity of the energy functional under field variations about $\omega$ implies that

$$(k-2)E(\omega)=0\,.$$

Thus, only in the principal chiral model, corresponding to $k=2$, is there any hope of finding non-trivial solutions compatible with the one-point compactification. These maps can be classified for example by the integer $n$ of the asymptotic behaviour $(1/r)^n$.

## 3. A strategy for solving the Gribov equation

The Euler-Lagrange equation (3) is a second order quasilinear elliptic system of partial differential equations. At present there is no systematic method of solving such equations, nor is there any existence theorem to apply in the present problem. (The existence and uniqueness theorems for the harmonic maps are known only in relatively few cases, and to the best of the author's knowledge, there are no results for harmonic maps from a non-compact manifold to a compact one which can be applied in the present problem.) In the lack of general results one is led to use various direct approaches. Unfortunately the most powerful methods, based on the properties of the perturbed energy functional [26] such as lower semicontinuity in suitable function spaces, are difficult to apply in attempting to solve the Gribov problem. This is a consequence of the divergence of the energy functional for critical maps in dimensions higher than two. This means that if one attempts to use the properties of the energy functional, one must first perform vacuum energy renormalizations. In the present problem this is a cumbersome approach and the ideas are obscured by mathematical peculiarities. Further, the power of this method is obvious only if one has some acceptable classification method of the allowed and physically interesting configurations at hand. With the lack of a suitable classification one is led to try finding methods that will yield interesting results by direct, albeit less strict, procedures.

The method that I will use is based on the splitting of the Euler-Lagrange equation (3) into simpler parts. After the subproblems have been solved one can construct harmonic maps from $\mathrm{R}^k$ to $\mathrm{SU}(N)$ and in this way obtain solutions to the original problem. The idea in the splitting used is the following: select some compact or non-compact submanifold M of $\mathrm{R}^k$. Every harmonic map $\omega$ from $\mathrm{R}^k$ to $\mathrm{SU}(N)$ induces a map from this submanifold M of $\mathrm{R}^k$ into some submanifold N of $\mathrm{SU}(N)$. The method is based on the construction of the trace map $\tilde{\omega}$: $\mathrm{M} \rightarrow \mathrm{N}$ of $\omega$ on the submanifold M, and after the trace map $\tilde{\omega}$ has been constructed, its immersion into the manifold $R^k$ provides a solution to the problem by giving the original map $\omega$. The choice of the submanifold $M \subset R^k$ is largely determined by the symmetry and long-distance behaviour one wants for the allowed field configurations, and the reasonable candidates for the submanifolds $N \subset SU(N)$ are the closed Lie subgroups of the group $SU(N)$ or the riemannian symmetric spaces, which are of the form $SU(N)/H$, with H a closed Lie subgroup of $SU(N)$. In practice the trace maps $\tilde{\omega} : M \to N$ are constructed by embedding harmonic maps $\hat{n}$, defined on the manifold M, into the manifold N. But I do not expect the harmonicity here to be an essential limitation if one wants the allowed field configurations to be compactifiable [compare eq. (7)]. The technical reason for limiting to such harmonic trace maps is that the immersion will be simple. Moreover, there might be important hints for studying non-linear $\sigma$-models on compact manifolds. The use of trace maps also provides a useful way of classifying the immersed maps: suppose that the manifold M is compact, as it often is if one assumes the WBC boundary condition (6). Then one can perform a homotopy classification for the trace maps $\tilde{\omega} : M \to N \simeq H$ or $SU(N)/H$, and the classification of the immersed maps can be given by using the homotopy classification of the trace maps $\tilde{\omega}$. The existence and uniqueness problem for harmonic maps between compact riemannian manifolds has been studied extensively [26] and in many cases one can give a complete classification of the harmonic maps.

## 4. Examples in the SU(2) gauge group

The group manifold SU(2) is homeomorphic to the euclidean sphere $S^3$. The condition that a map $\hat{n} : M \subset R^k \to S^{k'}$ is harmonic yields the equation

$$\partial^2 \hat{n} + (\partial_\mu \hat{n} \cdot \partial_\mu \hat{n})\hat{n} = 0 \,. \tag{11}$$

This equation says that the laplacian of the vector $\hat{n}$ is parallel to the vector $\hat{n}$ itself, and the coefficient of proportionality is the energy density $\partial_\mu \hat{n} \cdot \partial_\mu \hat{n}$. This is a general result: if we have the equation

$$\partial^2 \hat{n} + f\hat{n} = 0 \,, \tag{12}$$

with $f$ some function on $R^k$, one can find non-trivial solutions if and only if $f$ is the energy density $\partial_\mu \hat{n} \cdot \partial_\mu \hat{n}$. This can be seen by multiplying eq. (12) by $\hat{n}$ and solving for $f$. Thus eq. (11) is the most general equation with $\partial^2 \hat{n}$ and $\hat{n}$ parallel.

The homeomorphism $SU(2) \sim S^3$ suggests the use of spherical parametrization on the manifold SU(2). It is well known that the euclidean space $R^4$ admits two spherical coordinate systems [28, 29]: the normal system is given by

$$\begin{aligned} x &= r \sin\theta_1 \sin\theta_2 \sin\phi \,, \\ y &= r \sin\theta_1 \sin\theta_2 \cos\phi \,, \\ z &= r \sin\theta_1 \cos\theta_2 \,, \\ t &= r \cos\theta_1 \,, \end{aligned} \tag{13}$$

and the biharmonic system is defined by

$$
\begin{aligned}
x &= r \sin\theta \sin\phi_{12} , \\
y &= r \sin\theta \cos\phi_{12} , \\
z &= r \cos\theta \sin\phi_{34} , \\
t &= r \cos\theta \cos\phi_{34} .
\end{aligned} \tag{14}
$$

In order to keep the resulting equations tractable I will make use of both of these systems – the restrictions introduced in the simplified equations based on one system are avoided in the other system.

In the normal system (13) it is convenient to parametrize the gauge transform matrix $\omega$ by

$$\omega = \exp\{i\alpha\hat{n}\cdot\hat{\sigma}\} ,$$

where $\hat{n}\cdot\hat{n}=1$. Substituting this into the Euler-Lagrange equation (3) one gets the equations

$$\partial^2\alpha = \tfrac{1}{2}\sin 2\alpha\,\partial_\mu\hat{n}\cdot\partial_\mu\hat{n} , \tag{15}$$

$$\sin\alpha[\partial^2\hat{n}+(\partial_\mu\hat{n}\cdot\partial_\mu\hat{n})\hat{n}]+2\cos\alpha\,\partial_\mu\alpha\cdot\partial_\mu\hat{n}=0 . \tag{16}$$

Eqs. (15), (16) are useful especially if one tries to get solutions with the submanifold N of SU(2) being $S^2 \sim SU(2)/U(1)$ or SU(2) itself; assuming the orthogonality relation

$$\partial_\mu\alpha\,\partial_\mu\hat{n}=0 , \tag{17}$$

eqs. (15), (16) simplify to

$$\partial^2\alpha = \tfrac{1}{2}\sin 2\alpha\,\partial_\mu\hat{n}\cdot\partial_\mu\hat{n} , \tag{18}$$

$$\partial^2\hat{n}+(\partial_\mu\hat{n}\cdot\partial_\mu\hat{n})\hat{n}=0 . \tag{19}$$

Eq. (19) says that the trace map $\hat{n}: M\subset R^k \to S^2$ or $S^1$ is harmonic.

To enable the analysis of harmonic maps $\omega: R^k \to SU(2)$ without the restrictive assumption (17) I will give the equations based on the biharmonic system (14). In this case it is convenient to use the parametrization

$$\hat{\omega} = \sin\alpha\hat{n}_1 + \cos\alpha\hat{n}_2 ,$$

where the subsidiary conditions

$$\hat{n}_1\cdot\hat{n}_1 = \hat{n}_2\cdot\hat{n}_2 = 1 , \qquad \hat{n}_1\cdot\hat{n}_2 = 0 ,$$

for the two-vector trace maps $\hat{n}_1$ and $\hat{n}_2$ are assumed. The condition that $\hat{\omega}$ is a harmonic map yields the equations

$$\sin\alpha\,\partial^2\hat{n}_1+2\cos\alpha\,\partial_\mu\alpha\,\partial_\mu\hat{n}_1+\cos\alpha\,\partial^2\alpha\,\hat{n}_1$$
$$+\sin^3\alpha\,\partial_\mu\hat{n}_1\cdot\partial_\mu\hat{n}_1\hat{n}_1+\sin\alpha\,\cos^2\alpha\,\partial_\mu\hat{n}_2\cdot\partial_\mu\hat{n}_2\hat{n}_1=0\,, \tag{20}$$

$$\cos\alpha\,\partial^2\hat{n}_2-2\sin\alpha\,\partial_\mu\alpha\,\partial_\mu\hat{n}_2-\sin\alpha\,\partial^2\alpha\,\hat{n}_2$$
$$+\cos^3\alpha\,\partial_\mu\hat{n}_2\cdot\partial_\mu\hat{n}_2\hat{n}_2+\cos\alpha\,\sin^2\alpha\,\partial_\mu\hat{n}_1\cdot\partial_\mu\hat{n}_1\hat{n}_2=0\,. \tag{21}$$

In order to keep the immersion simple I will here assume the orthogonality conditions

$$\partial_\mu\alpha\,\partial_\mu\hat{n}_1=0\,,\qquad \partial_\mu\alpha\,\partial_\mu\hat{n}_2=0\,.$$

These equations imply that the trace maps $\hat{n}_1$ and $\hat{n}_2$ are parallel to their laplacians $\partial^2\hat{n}_1$ and $\partial^2\hat{n}_2$, respectively, and we know that this implies that the trace maps $\hat{n}_1$ and $\hat{n}_2$ are harmonic maps. In the present case they are harmonic maps from the submanifold $\mathrm{M}\subset\mathrm{R}^k$ to the circle $\mathrm{S}^1$, so that the submanifold N of the gauge group SU(2) will be the riemannian symmetric space SU(2)/U(1) or the gauge group SU(2) itself. Eqs. (20) and (21) now simplify to

$$\partial^2\alpha=\tfrac{1}{2}\sin 2\alpha[\partial_\mu\hat{n}_1\cdot\partial_\mu\hat{n}_1-\partial_\mu\hat{n}_2\cdot\partial_\mu\hat{n}_2]\,, \tag{22}$$

$$\partial^2\hat{n}_1+(\partial_\mu\hat{n}_1\cdot\partial_\mu\hat{n}_1)\hat{n}_1=0\,, \tag{23}$$

$$\partial^2\hat{n}_2+(\partial_\mu\hat{n}_2\cdot\partial_\mu\hat{n}_2)\hat{n}_2=0\,. \tag{24}$$

These equations should be compared with the eqs. (18) and (19).

I will now discuss some examples for the equations presented. First I will discuss vacuum copies arising from the normal system equations (18) and (19).

The first case to be analyzed corresponds to harmonic trace maps $\hat{n}$ from some submanifold M of $\mathrm{R}^k$ to the circle $\mathrm{S}^1$. The circle $\mathrm{S}^1$ will be embedded as a sphere $\mathrm{S}^2\simeq\mathrm{SU(2)/U(1)}$ to the gauge group SU(2). The vacuum copies arising from this splitting have the interesting property that the topological flux [4],

$$q(t_0)=\frac{1}{4\pi^2}\,\varepsilon_{ijk}\varepsilon_{abc}\int_{t=t_0}\partial_i\alpha\,\sin^2\alpha\,n^a\,\partial_j n^b\,\partial_k n^c\,\mathrm{d}^3x\,, \tag{25}$$

vanishes for them. This can be seen directly from eq. (25). Thus all solutions in this family correspond to the topological state characterized by the trivial configuration $A_\mu\equiv 0$.

Assume that the trace map $\hat{n}$ in eq. (19) sends some submanifold $\mathrm{M}\subset\mathrm{R}^k$ to the circle $\mathrm{S}^1$. This can be adjusted by choosing $\hat{n}$ to be

$$\hat{n}=\hat{A}\begin{bmatrix}\sin\phi\\ \cos\phi\\ 0\end{bmatrix},$$

where $\hat{A}$ is a constant orthogonal matrix and $\phi$ is now a real valued function on the manifold $M \subset R^k$. The trace map $\hat{n}$ is harmonic if the function $\phi$ is a harmonic function on $R^k$,

$$\partial^2 \phi = 0 .$$

The orthogonality condition (17) now gets the form

$$\partial_\mu \alpha \, \partial_\mu \phi = 0 ,$$

and if one wants the immersion in terms of an ordinary differential equation for $\alpha$, one must require that $\partial_\mu \phi \, \partial_\mu \phi$ depends only on the same variable as $\alpha$. Solutions are easily found and I will present one. The solution has previously been presented in ref. [5]: assume that $\phi$ is given by

$$\phi(x, y, z, t) = \omega t ,$$

which gives $\partial_\mu \phi \, \partial_\mu \phi = \omega^2$. If $\alpha$ now depends on $u$,

$$u = \frac{\omega(ax + by + cz)}{\sqrt{a^2 + b^2 + c^2}} ,$$

the immersion is provided by solutions to the equation of the mathematical pendulum,

$$\ddot{\alpha} = \tfrac{1}{2} \sin 2\alpha . \tag{26}$$

This equation can be solved in terms of elliptic functions and the solution corresponding to the oscillation from the unstable position of equilibrium, $\alpha(t = -\infty) = 0 + n\pi$ to the unstable position of equilibrium, $\alpha(t = +\infty) = 0 + (n \pm 1)\pi$, is given by

$$\alpha(u) = \pm 2 \arctan [\exp \{\sqrt{2} u + c\}) .$$

The other interesting riemannian symmetric space N of the gauge group SU(2) is the manifold SU(2) itself. This case arises by embedding the trace maps $\hat{n} : M \subset R^k \to S^2 \simeq CP^1$, and has been studied comprehensively in the literature. I will first give the equations for a harmonic map $\hat{n} : M \to S^2$,

$$\hat{n} = \begin{bmatrix} \sin \lambda \sin \zeta \\ \cos \lambda \sin \zeta \\ \cos \zeta \end{bmatrix} .$$

Here $\lambda$ and $\zeta$ are real valued functions on M. The harmonicity condition (11) gives

$$\partial^2 \zeta = \tfrac{1}{2} \sin 2\zeta \partial_\mu \lambda \, \partial_\mu \lambda , \qquad \partial_\mu (\sin^2 \zeta \partial_\mu \lambda) = 0 .$$

Solutions to these equations are easily found: for example choosing $M = R^2$ and requiring finiteness of the energy functional (4) the conformal invariance of the energy functional can be used and the most general solution is given by the $CP^1$

multi-instanton and multi-anti-instanton solutions (see for example ref. [25]). On the Riemann sphere these solutions are given by the map $z \to z^k$ and the maps complex conjugate to this. The integer $k$ is the Brouwer degree of the map and equals the instanton number. The identity map $z \to z$ corresponding to the one-instanton solution can be presented by a harmonic polynomial map by mapping $R^2$ stereographically onto the unit sphere $S^2 \subset R^3$. The result is

$$\hat{n} = \begin{bmatrix} \sin\phi \sin\theta \\ \cos\phi \sin\theta \\ \cos\theta \end{bmatrix}. \tag{27}$$

The notation refers to the spherical system on $R^3$. The anti-instanton will yield the antipodal map which is given by the map (27) by changing $\phi \to -\phi$. I will now prove that the identity and antipodal maps are the only non-trivial harmonic maps $S^2 \to S^2$ that can be immersed consistently with condition (17). Note that according to eq. (7) every trace map is asymptotically a harmonic map between spheres. For this purpose I will stereographically project the $CP^1$ multi-instanton solutions onto the unit sphere $S^2$. The resulting map is given by [30]

$$\hat{n} = \begin{bmatrix} \sin k\phi \sin\{2\arctan[c(\tan\frac{1}{2}\theta)^{\pm k}]\} \\ \cos k\phi \sin\{2\arctan[c(\tan\frac{1}{2}\theta)^{\pm k}]\} \\ \cos\{2\arctan[c(\tan\frac{1}{2}\theta)^{\pm k}]\} \end{bmatrix}, \tag{28}$$

where $k$ is the instanton (or anti-instanton) number and $c>0$ is a parameter. From this result it can be seen that the vectors $\hat{n}$, $\partial\hat{n}/\partial\theta$ and $\partial\hat{n}/\partial\phi$ are mutually orthogonal. The orthogonality condition (17) can be expressed in the form

$$\frac{\partial\alpha}{\partial\theta}\frac{\partial\hat{n}}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial\alpha}{\partial\phi}\frac{\partial\hat{n}}{\partial\phi} = 0\,.$$

But, by the linear independence of the vectors $\partial\hat{n}/\partial\theta$ and $\partial\hat{n}/\partial\phi$ this is possible if and only if

$$\frac{\partial\alpha}{\partial\theta} = \frac{\partial\alpha}{\partial\phi} = 0\,,$$

which means that the immersion function $\alpha$ cannot depend on the angles $\theta$ and $\phi$. The energy density for the $CP^1$ multi-instanton solutions has the form

$$\partial_\mu\hat{n}\cdot\partial_\mu\hat{n} = \frac{1}{r^2}\frac{8k^2c^2\sin^{\pm 2k}\frac{1}{2}\theta\cos^{\pm 2k}\frac{1}{2}\theta}{\sin^2\theta[\cos^{\pm 2k}\frac{1}{2}\theta + c^2\sin^{\pm 2k}\frac{1}{2}\theta]^2}, \tag{29}$$

which is independent of the angles $\theta$, $\phi$ if and only if $k=\pm 1$, $c=1$ or $k=0$, $c>0$ corresponding to the identity, antipodal and trivial maps, respectively. Substituting the energy density (29) into the immersion equation (18) and noticing that $\alpha$ cannot depend on the angles $\theta$ and $\phi$ one gets the desired uniqueness result for the Gribov solution. The interesting feature of this result is that in the homotopy classes of the identity and antipodal maps there is a continuous family of harmonic maps, parametrized by the number $c > 0$. But only if $c = 1$, can the trace maps be immersed.

In the Landau gauge, choosing the manifold $M \subset R^4$ to be the sphere $S^3$, the trace map $\hat{n}$ in eq. (19) will be a harmonic map $S^3 \to S^2$. The homotopy structure of these maps is isomorphic to the additive group of integers and the existence problem for harmonic maps in these homotopy classes has been given a partial answer [31]: the Hopf fibration $S^3 \to S^2$ generates the homotopy structure and in the biharmonic representation it has the form [7]

$$\hat{n} = \begin{bmatrix} \sin(\phi_{12}+\phi_{34}) \sin 2\theta \\ \cos(\phi_{12}+\phi_{34}) \sin 2\theta \\ \cos 2\theta \end{bmatrix}.$$

This is a harmonic polynomial map. By joining the Hopf fibration and the $CP^1$ multi-instanton solution one gets harmonic maps $S^3 \to S^2$ in the homotopy classes with Hopf invariant of the form $\pm k^2$. The existence problem for harmonic maps in the other homotopy classes is not solved; however, it seems that the Hopf fibration and the corresponding antipodal map are the only harmonic polynomial maps in this case. The existence problem for the immersion equation (18) has been given an affirmative answer in the case of harmonic homogeneous polynomial maps [7], and for non-homogeneous polynomial harmonic maps one can perform a non-existence analysis similar to that in the Coulomb gauge.

The example to be presented next arises from the biharmonic system equations and thus avoids the orthogonality condition (17). The solution will satisfy the Landau gauge-fixing condition.

Let us choose the submanifold $M \subset R^4$ to be the torus $T^2 = S^1 \times S^1$ and the submanifold N to be the group SU(2) itself. This is achieved by the trace maps

$$\begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix} = \begin{bmatrix} \sin(k_1\phi_{12}+l_1\phi_{34}) \\ \cos(k_1\phi_{12}+l_1\phi_{34}) \\ \sin(k_2\phi_{12}+l_2\phi_{34}) \\ \cos(k_2\phi_{12}+l_2\phi_{34}) \end{bmatrix}, \tag{30}$$

where the notation refers to the biharmonic system (14) with $\rho_1 = r \sin\theta$ and $\rho_2 = r\cos\theta$. The immersion of the trace maps are given by solutions of the equation

$$\frac{\partial^2\alpha}{\partial\rho_1^2} + \frac{1}{\rho_1}\frac{\partial\alpha}{\partial\rho_1} + \frac{\partial^2\alpha}{\partial\rho_2^2} + \frac{1}{\rho_2}\frac{\partial\alpha}{\partial\rho_2} + \frac{1}{2}\left(\frac{k_2^2-k_1^2}{\rho_1^2} + \frac{l_2^2-l_1^2}{\rho_2^2}\right)\sin 2\alpha = 0\,, \tag{31}$$

where I have assumed that the immersion function $\alpha$ depends on the variables $\rho_1$ and $\rho_2$ only. Real analyticity at the origin will limit the allowed values of $k_2^2 - k_1^2$ and $l_2^2 - l_1^2$ so that $\sqrt{|k_2^2-k_1^2|}$ and $\sqrt{|l_2^2-l_1^2|}$ are integers. Moreover, the inequality $k_2^2 < k_1^2$ is possible if and only if $l_2^2$ is smaller than $l_1^2$. In the following I restrict the analysis to the case $k_2^2 < k_1^2$ as the other cases can be analyzed similarly. In order to get eq. (31) to a simpler form I assume that the immersion function $\alpha$ depends only on the variable

$$t = \sqrt{k_1^2 - k_2^2}\ln \rho_1 + \sqrt{l_1^2 - l_2^2}\ln \rho_2\,,$$

which, upon substituting into eq. (31), will give the equation of the mathematical pendulum,

$$\ddot{\alpha} = \tfrac{1}{2}\sin 2\alpha\,,$$

already found in our earlier analysis, eq. (26). The solution presented is, however, in general a weak solution (a solution in the sense of distributions) due to the "angular singularity" at the origin, and in order to get classical solutions we set $k_2 = l_2 = 0$. By substituting $t = \ln(\rho_1/\rho_2)$ into eq. (31) one arrives at a family of solutions similar to that presented in ref. [8]. However, from the preceding analysis it should be clear that these solutions are solutions only in the weak sense, and the field strength tensor will be a distribution with a support concentrated at the origin: in particular, these maps will not yield vacuum copies on $\mathrm{R}^4$ but on $\mathrm{R}^4 - \{0\}$.

## 5. Examples in the SU(3) gauge group

In the SU($N$) case one can always choose some SU(2) subgroup of the gauge group SU($N$) and in this way the solutions of the SU(2) gauge group provide solutions in the general SU($N$) gauge group as well. But from the physical and mathematical point of view it is of interest to study the new features introduced when $N$ is larger than two.

The parametrization of the general SU($N$) gauge group element cannot be given such a simple form as in the case of SU(2). Thus the immersion of the submanifold $\mathrm{M} \subset \mathrm{R}^k$ will, in general, be extremely complicated, and some kind of strategy is needed. For this purpose I will briefly present the relevant substructures of the gauge group SU($N$) [32]. The first substructures that come to mind are the closed Lie subgroups H of the gauge group SU($N$). The less trivial substructures of interest in the present problem are the riemannian symmetric spaces S, which are of the form $\mathrm{S} \simeq \mathrm{SU}(N)/\mathrm{H}$. The riemannian symmetric space S can be embedded into the gauge group manifold SU($N$) as a closed, totally geodesic submanifold, and the invariant vector field $\underline{\mathrm{su}}(N)$ on the manifold SU($N$) can be decomposed into a vertical subspace $\underline{\mathrm{h}}$, which is isomorphic to the Lie algebra of the subgroup H, and its orthogonal complement $\underline{\mathrm{s}}$,

$$\underline{\mathrm{su}}(N) = \underline{\mathrm{h}} + \underline{\mathrm{s}}\,.$$

The Killing form on $\underline{\mathrm{su}}(N)$ and its restriction to $\underline{\mathrm{s}}$ defines a unique SU($N$) invariant riemannian structure on the manifolds SU($N$) and S, respectively, and the restriction of the exponential map to the vector field $\underline{\mathrm{s}}$ is locally homeomorphic to the riemannian symmetric space S. The relevant commutator relations of the vector fields $\underline{h}$ and $\underline{s}$ are

$$[\underline{h}, \underline{h}] \subset \underline{h}\,, \qquad [\underline{h}, \underline{s}] \subset \underline{s}\,, \qquad [\underline{s}, \underline{s}] \subset \underline{h}\,.$$

When $N \geq 3$, it is in general very difficult to find explicit harmonic maps from $R^k$ to SU($N$), and all explicit constructions [10, 11] have been based on constructing harmonic maps from $R^k$ to some subgroup or riemannian symmetric space of the gauge group SU($N$): the harmonic trace maps are embedded into the tangent spaces $\underline{h}$ or $\underline{s}$, and, by immersing these trace maps into $R^k$, vacuum copies have been found. Harmonic maps into the submanifolds of SU($N$) can be constructed easier by utilizing the properties of the submanifolds, and in searching for vacuum copies with some desired spatial symmetry the use of Euler-Lagrange equations can be avoided by applying the general principle by Coleman and Faddeev [33, 34]. In the present case the argumentation proceeds roughly as follows: in looking for harmonic maps invariant under some spatial symmetry group G one must only make the energy functional (4) stationary for the most general field configuration invariant under G as the energy functional is automatically stationary under asymmetric variations, as a consequence of Schur's lemma. Utilizing the subgroups or riemannian symmetric spaces embedded into the group manifold SU($N$) as closed, totally geodesic submanifolds, further simplification can be obtained by restricting the symmetric field configuration to be varied so that it has values only in the chosen submanifold N of SU($N$). In this way the resulting stationary field configuration will be a harmonic map from $R^k$ to N $\subset$ SU($N$) automatically solving the Euler–Lagrange equation (3). By choosing the submanifold N suitably the variation of the general configuration invariant under the spatial symmetry group G will be relatively simple and the existence of solutions to the immersion equation can be proved directly.

Before presenting explicit solutions in the SU(3) case, I will mention some solutions available in related fields of study.

The use of subgroups or riemannian symmetric spaces (or some other totally geodesic spaces) of the gauge group SU($N$) will prove useful by recalling the commutator equation (5). For example, the riemannian symmetric space $CP^{N-1}$ can be embedded into the group manifold SU($N$), $CP^{N-1} \simeq SU(N)/SU(N-1)$, and by multiplying two $CP^{N-1}$ multi-instanton solutions [35, 36] living in the different $R^2$ parts of $R^4$, vacuum copies satisfying the Landau gauge-fixing condition will arise. Of course a $CP^{N-2}$ multi-instanton solution, taking values for example in the subgroup SU($N-1$) can be chosen as the other partner. There are also many other non-linear $\sigma$-models, solutions of which can be used in this context: for example the O($2N$) solutions [37] can be chosen in connection with the riemannian symmetric spaces $S^{2N-1} \simeq SU(N)/SU(N-1)$. All the finite-action solutions of the 2-dimensional non-linear $\sigma$-models live in the compact space $R^2 \cup \{\infty\} \simeq S^2$ and so the corresponding vacuum copies will live in the compact space $S^2 \times S^2$. And by noticing that the energy functional is invariant under conformal transforms of the domain in two dimensions, the construction of solutions to the commutator equation (5) can be reduced to the problem of constructing harmonic maps between the compact manifolds $S^2$ and $SU(N)$. By expressing the Euler–Lagrange equation (3) in local coordinates on the manifolds $R^k$ and $SU(N)$ it will take the form

$$\Delta\omega_\alpha + \frac{\partial\omega_\beta}{\partial x_i}\frac{\partial\omega_\gamma}{\partial x_i}\Gamma_\alpha^{\beta\gamma} = 0 .$$

Here $\Delta$ is the laplacian on $R^k$ and the $\Gamma_\alpha^{\beta\gamma}$'s are the Christoffel symbols on $SU(N)$. In the case of $R^1$ this equation reduces to the equation of geodetic lines on $SU(N)$, and by splitting $R^k = R^1 \times \cdots \times R^1$ and multiplying $k$ geodesics living in the different $R^1$ parts, solutions to the commutator equation (5) can be constructed. Naturally, mixed solutions of the type $R^1 \times \cdots \times R^2 \times \cdots \times R^k$ can also be presented. In fact, the solution presented in ref. [6] is actually a solution arising from the splitting $R^1 \times R^2$ with the $R^2$ part represented by the $CP^1$ instanton solution.

I will not carry on this formal discussion of the general gauge group $SU(N)$ further, but show how the ideas for constructing explicit copies will work in the case of SU(3) by giving two examples. In order to find useful substructures I will first review some well-known properties of the group SU(3).

An SU(2) subgroup of SU(3) can be constructed by using the Gell–Mann matrices $\lambda_1, \lambda_2, \lambda_3$. One can also use the matrices $\lambda_4, \lambda_5, \frac{1}{2}\lambda_3 + \frac{1}{2}\sqrt{3}\lambda_8$ or the matrices $\lambda_6$, $\lambda_7$, $-\frac{1}{2}\lambda_3 + \frac{1}{2}\sqrt{3}\lambda_8$ to form an SU(2) subgroup. These subgroups can be enlarged by noticing for example that $\lambda_8$ commutes with the matrices $\lambda_1, \lambda_2, \lambda_3$. This set of four Gell–Mann matrices generates the subgroup $SU(2) \times U(1)$. A new subgroup can be constructed for example from the SO(3) generators $\lambda_2$, $\lambda_5$, $\lambda_7$. These and other subgroups can be used in constructing riemannian symmetric spaces. For example, the complex projective space $CP^2$ is given as the coset space $SU(3)/SU(2) \times U(1)$, the sphere $S^5$ can be given as the symmetric space $SU(3)/SU(2)$ and the subgroup SO(3) will yield the unitary symmetric space $SU(3)/SO(3)$.

As the first example I construct harmonic maps from $R^3$ and $R^4$ to the geodesic sphere $S^5 \simeq SU(3)/SU(2)$ embedded in the manifold SU(3). The method presented is, however, by no means restricted to the sphere $SU(3)/SU(2)$ but can also be applied in the general case of $S^{2N-1} \simeq SU(N)/SU(N-1)$.

The parametrization of $S^5$ is most conveniently given by some spherical coordinate system in $R^6$ where I have now embedded the manifold $S^5 \simeq SU(3)/SU(2)$ as a unit sphere. There are six spherical systems in $R^6$ [28, 29] and all of them can be used succesfully in constructing harmonic maps. As an example of the construction method I will employ the normal system, and the harmonic map $\hat{\omega} : M \subset R^k \to S^5 \subset R^6$ is parametrized as follows:

$$\hat{\omega} = (\sin\alpha\hat{n}, \cos\alpha) .$$

Here $\hat{n}$ is a harmonic trace map into $S^4 \subset S^5$ and $\alpha$ is the immersion function. Immersion of the trace map will yield eq. (18). I will limit the handling to the polynomial trace maps $\hat{n} : S^{k-1} \subset R^k \to S^4 \subset S^5$, which suggests that solutions to the immersion equation cannot be found unless $\alpha$ depends on the radial variable $r$ only.

In the case of $R^3$ corresponding to the Coulomb gauge, the unique [38] harmonic polynomial map $\hat{n}: S^2 \to S^4$ which cannot be interpreted as a map to some lower dimensional sphere in $S^4$ is given by

$$\hat{n} = \begin{bmatrix} \frac{3}{2}\cos^2\theta - \frac{1}{2} \\ \frac{1}{2}\sqrt{3}\sin\phi\sin 2\theta \\ \frac{1}{2}\sqrt{3}\cos\phi\sin 2\theta \\ \frac{1}{2}\sqrt{3}\sin 2\phi\sin^2\theta \\ \frac{1}{2}\sqrt{3}\cos 2\phi\sin^2\theta \end{bmatrix}.$$

By substituting $t = \ln r$ into the immersion equation, one gets the Gribov pendulum equation [1, 2] in the form

$$\ddot{\alpha} + \dot{\alpha} = 3\sin 2\alpha,$$

which concludes the construction.

In the Landau gauge one can use the methods presented in ref. [7] in constructing the trace maps and the harmonic polynomial trace map $\hat{n}: S^3 \to S^4$ can be taken to be, for example,

$$\hat{n} = \begin{bmatrix} \frac{1}{2}\sin(\phi_{12}+\phi_{34})\sin 2\theta \\ \frac{1}{2}\cos(\phi_{12}+\phi_{34})\sin 2\theta \\ \frac{1}{2}\sin(\phi_{12}-\phi_{34})\sin 2\theta \\ \frac{1}{2}\cos(\phi_{12}-\phi_{34})\sin 2\theta \\ \cos 2\theta \end{bmatrix}.$$

The notation refers to the biharmonic system (14). By substituting $t = 2\ln r$ into the immersion equation we get the Gribov pendulum equation

$$\ddot{\alpha} + \dot{\alpha} = \sin 2\alpha.$$

Another choice of harmonic trace maps would be

$$\hat{n} = \begin{bmatrix} \frac{3}{2}\cos^2 2\theta - \frac{1}{2} \\ \frac{1}{2}\sqrt{3}\sin(\phi_{12}\pm\phi_{34})\sin 4\theta \\ \frac{1}{2}\sqrt{3}\cos(\phi_{12}\pm\phi_{34})\sin 4\theta \\ \frac{1}{2}\sqrt{3}\sin(2\phi_{12}\pm 2\phi_{34})\sin^2 2\theta \\ \frac{1}{2}\sqrt{3}\cos(2\phi_{12}\pm 2\phi_{34})\sin^2 2\theta \end{bmatrix},$$

and by substituting $t=2\ln r$ the immersion equation will yield the Gribov pendulum equation

$$\ddot{\alpha}+\dot{\alpha}=3\sin 2\alpha .$$

The maps constructed are harmonic maps from the euclidean space $R^k$ to the sphere $S^5$ which can be realized as a totally geodesic space in the gauge group manifold SU(3). Thus, the maps constructed will give vacuum copies in the SU(3) gauge group.

The next example is an application of the general principle of Coleman and Faddeev in the Coulomb gauge. The subgroup G of the full symmetry group is chosen to be the group of simultaneous rotations in the spaces $R^3$ and SU(3), i.e., rotations generated by $\hat{L}+\hat{T}$ where $\hat{L}$ is the three-dimensional angular momentum and $\hat{T}$ is the generator of rotations in the configuration space. The harmonic maps $\hat{\omega}: R^3 \to$ SU(3) that we are seeking for can thus be characterized by the commutator relation

$$[\hat{L}+\hat{T}, \hat{\omega}]=0 .$$

The most general map $\hat{\omega}$ invariant under the combined $SO(3)\times SO(3)$ action of a rotation in the base space $R^3$ and a global similarity transform in the configuration space SU(3) is of the form

$$\hat{\omega}_{ik}=a(r)\phi_{ik}+b(r)\psi_{ik}+c(r)\delta_{ik} ,$$

where $a$, $b$ and $c$ are some complex valued functions of the radial variable $r$ and the matrices $\phi$, $\psi$ are defined by

$$\phi_{ik}=n_in_k-\tfrac{1}{3}\delta_{ik}=n_1n_2\lambda_1+n_1n_3\lambda_4+n_2n_3\lambda_6+\tfrac{1}{2}(n_1^2-n_2^2)\lambda_3+\tfrac{1}{2}\sqrt{3}(\tfrac{1}{3}-n_3^2)\lambda_8 ,$$

$$\psi_{ik}=-i\varepsilon_{ikl}n_l=n_1\lambda_7-n_2\lambda_5+n_3\lambda_2 .$$

The unit vector $\hat{n}$ is the one given in eq. (27). By setting $n_1=1$, $n_2=n_3=0$ (note that $a$, $b$, $c$ are assumed to depend on the variable $r$ only) and making use of the unitarity of $\hat{\omega}\in$ SU(3) and noticing that det $(\hat{\omega})=1$, one gets the equations

$$(\tfrac{2}{3}a+c)(\tfrac{2}{3}\bar{a}+\bar{c})=1 ,$$

$$(-\tfrac{1}{3}a+c)(-\tfrac{1}{3}\bar{a}+\bar{c})+b\bar{b}=1 ,$$

$$(-\tfrac{1}{3}a+c)\bar{b}+(-\tfrac{1}{3}\bar{a}+\bar{c})b=0 ,$$

$$(-\tfrac{1}{3}a+c)^2(\tfrac{2}{3}a+c)-b^2(\tfrac{2}{3}a+c)=1 .$$

By a straightforward calculation based on these equations one can now show that the map $\hat{\omega}$ can be given the form

$$\hat{\omega}=\exp(i\alpha(r)\phi)\exp(i\beta(r)\psi) , \tag{32}$$

where $\alpha$ and $\beta$ are arbitrary real valued functions depending on the radial variable $r$ only. This is exactly the trial map used in ref. [10], and by substituting (32) into the energy functional (4) and performing the variation one gets the Euler-Lagrange equations

$$\frac{\partial^2\alpha}{\partial r^2}+\frac{2}{r}\frac{\partial\alpha}{\partial r}=\frac{6}{r^2}\sin\alpha\cos\beta\,,$$

$$\frac{\partial^2\beta}{\partial r^2}+\frac{2}{r}\frac{\partial\beta}{\partial r}=\frac{2}{r}\sin\beta\cos\alpha\,.$$

By substituting $t=\ln r$ these immersion equations can be put into the Yee-Viswanathan form

$$\ddot{\alpha}+\dot{\alpha}=6\sin\alpha\cos\beta\,, \tag{33}$$

$$\ddot{\beta}+\dot{\beta}=2\sin\beta\cos\alpha\,. \tag{34}$$

If one sets $\beta=0$ the pure SU(3)/SO(3) immersion solutions are found. These solutions are characterized by the asymptotic behaviours $\alpha(-\infty)=0$, $\alpha(+\infty)=\pm\pi$ (modulo $2\pi$). The pure SO(3) immersion solutions $\alpha\equiv0$ have the asymptotics $\beta(-\infty)=0$, $\beta(+\infty)=\pm\pi$. Besides these solutions there are also mixed solutions. These solutions can be analyzed by the mechanical analogue of eqs. (33) and (34) describing a pendulum acted on by a periodic force and moving under the influence of some kind of a viscous friction. A numerical analysis of the equations shows that there are three different types of solution: first, the pendulum starts from $\alpha(-\infty)=\beta(-\infty)=0$ and rolls down the potential surface and comes to rest at the minimum points $\alpha(+\infty)=\pm\pi$, $\beta(+\infty)=0$. These solutions belong to the SU(3)/SO(3) family. The solutions of the SO(3) family are characterized by the asymptotic behaviour $\alpha(+\infty)=0$, $\beta(+\infty)=\pm\pi$. A typical numerical integration of these kind of solutions is given in fig. 1. The third type of solution corresponds to separatrixes characterized by

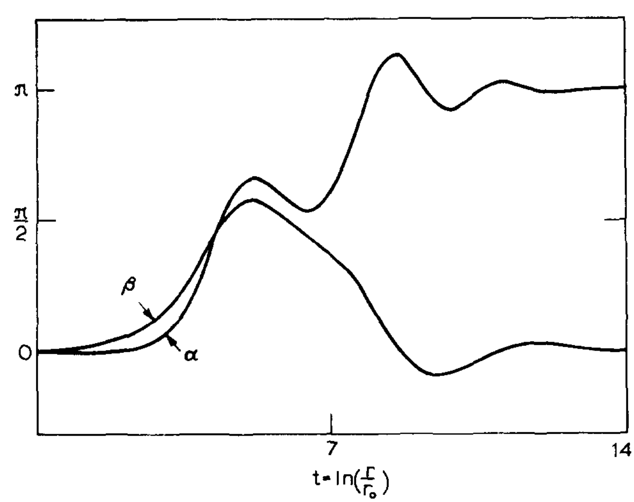


Fig. 1. A numerical integration of the Yee-Viswanathan system corresponding to a solution with line topology.

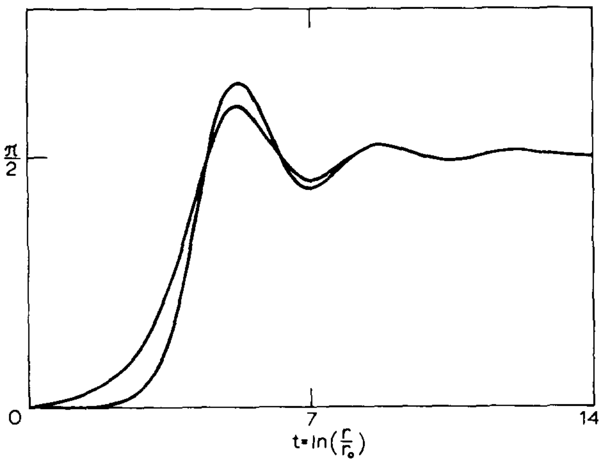


Fig. 2. A numerical integration of the Yee–Viswanathan system corresponding to a solution with plane topology.

the asymptotic behaviours $\alpha(+\infty) = \pm\frac{1}{2}\pi$, $\beta(+\infty) = \pm\frac{1}{2}\pi$. A numerical integration corresponding to this kind of a solution is presented in fig. 2. I have performed a thorough numerical analysis of eqs. (33) and (34) and it seems that the viscous friction is so large that no other solutions are possible in the present context.

## 6. Discussion

I have presented a systematic method for solving the Gribov vacuum copy equation in the SU($N$) Coulomb and Landau gauges and given many examples of how the method can be used.

Even though there now is some kind of system for solving the Gribov equation, the problem itself, Gribov ambiguities, has not yet been given a reasonable physical interpretation. The mere existence of gauge-fixing conditions that do not allow Gribov copies [12, 13] shows that the physical interpretation is not easy to find: the spatial part of the Minkowski space, $\mathrm{R}^{1,3}$, is the euclidean three-space $\mathrm{R}^3$ which is non-compact. In studying physical processes of real particles one can always eliminate the extra degrees of freedom from the classical lagrangian by choosing some complete, continuous gauge-fixing condition such as the Cronström gauge [13],

$$x^\mu A_\mu = 0 .$$

Thus one can argue that the Gribov ambiguities are irrelevant from the physical point of view. The situation is, however, drastically chanced if one is interested in the internal structure of hadrons: if one is viewing the hadron from inside, as a quark and gluon would, one must take into account the boundary conditions for the gluon fields on the boundary of the hadron. Mathematically this means that the function space of the allowed field configurations is limited and the hadron will be a Stone-Chech like compactified topological space, possibly with some substructures. In the internal world of the hadron this means that the gluons will feel that they live in a compactified space. But in a suitably compactified space one cannot avoid Gribov ambiguities [14, 15] which means that the spectrum of states inside the hadron does not contain free gluon states. This means that color is confined. Note that any trial to avoid gauge degeneracy by choosing non-continuous gauge-fixing conditions means that in the quantized theory the gauge is fixed by some operator condition; and the gauge fixing, when operating in the Hilbert space of physical states, is ill defined for some states, which is not very pleasant.

The choice of boundary conditions at infinity depends on the picture one is using. For example, in the magnetic string model of hadrons the quarks must have magnetic charges which are the sources of the flux along the string. Thus, the choice of boundary conditions must be consistent with the long-range behaviour of magnetic monopoles. At present there are several boundary conditions that have been applied but one has no clear way to choose the right compactification.

At this moment it is of interest to recall the uniqueness of the gauge fixing in an abelian theory like QED. Here the gauge transform matrix is an ordinary function,

$$\omega = e^{i\phi}$$

with $\phi$ a real valued function. The uniqueness of the Coulomb and Landau gauge-fixing condition says that $\phi$ is a harmonic function,

$$\partial^2\phi = 0\,.$$

But the solution $\phi = \text{const.}$, is the only harmonic function on $R^3$ or $R^4$ that is consistent with some compactification. Thus one can, inside the hadron, always exclude the extra variables from the lagrangian and the photons can be realized in the spectrum of states inside the hadron. But the spectrum of states is a Hilbert space which is a linear vector space and so the hadron can emit the photons: no confining effect is possible.

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